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Orthonormal Tetrad Transport. II. Physical Identification of the Normals and Curvatures of a Time-Like Curve

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Abstract

A flat space-time time-like curve is considered from the point of view of an instantaneous comoving inertial observer. In the context of the 'vierbein' formalism a projection operator is introduced, able to project 3-vectors, belonging to the 3-space of the comoving observer, out of space-time 4-vectors. The motion of an accelerated particle relative to the comoving inertial frame is briefly reviewed by means of the projection technique, and the three space-like components of the Frenet-Serret tetrad are thus projected, and their motion relative to the comoving observer neatly stated. Finally, the physical identification of the normals and the curvatures obtains in terms of three-dimensional kinematics as seen by the instantaneous comoving observer.

1. *Introduction*

In a previous paper (Krause, 1974), hereafter referred to as I, some kinematic features of the 'vierbein' formalism were discussed and the process of Frenet-Serret transport was introduced. Space-time differential geometry associates with each event of a time-like world-line three particularly interesting kinds of orthonormal tetrads; namely, the instantaneous comoving free-falling tetrads, the Fermi-Walker transported tetrads, and the Frenet-Serret tetrad. In this paper we shall mainly be interested in this last structure, which we will think of as a local reference frame of some accelerated observer. Accordingly, an interesting problem in relativistic kinematics is to elucidate the physical meaning of the curvatures and the normals of a given time-like world-line.‡ (By 'physical meaning' we here understand how we can describe these four-

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‡ This problem seems to have been neglected in the literature. To this author's knowledge there is one exception, however. Synge has thrown much light on this question, being able to obtain the physical identification of the normals and the curvatures of the world-line of an observer on the earth in the general relativistic context; cf. Synge (1964).

dimensional geometric structures in terms of everyday three-dimensional kinematics.)

It is well established today that the best (if not the only) approach to relativity theory is attained by considering it as geometry of absolute space-time. Thus, the principle of general relativity requires that the laws of physics, dealing with events occurring in space in the course of time, have to conform to this geometric approach in a generally covariant manner. Furthermore, because of the lack of a universal Newtonian time, relativity theory imposes, in a very fundamental way, the four-dimensional description of material phenomena. Certainly, the four-dimensional formulation, besides being the most elegant way of expressing the theory in mathematical language, is the only known way of fully developing the idea that space and time are intimately melt forming a Riemannian 4-space. As geometers, we have to emphasise this point as strongly as possible. However, as physicists, we have to stress at every step of our analysis the deep difference between space and time, which becomes somewhat concealed by the four-dimensional tensor language of geometry. Therefore, in every relativistic problem whose geometric solution we have been lucky to grasp, we should be able to introduce a 'reasonable' three-dimensional point of view, somehow leading to a description in terms of 'ordinary' concepts, so we can grasp the physical content of the theory, at least, locally. It is in accordance with this spirit of real physical understanding that we want to elucidate in this paper what are the physical observables hidden behind the above-mentioned Frenet-Serret four-dimensional structure.

In the prèsent work we develop our theme in the context of the special theory of relativity since, clearly, the 'vierbein' projection manipulations can be more neatly understood in flat space-time. In general relativity the physical interpretation one gets of the Frenet-Serret structure is not so 'neat' because of the unavoidable approximations needed in curved space-time. Of course, the whole formalism discussed in this paper still works in the general theory, but becomes much more cumbersome to handle.

2. *The Frenet-Serret Observer and the Instantaneous Comoving Inertial Observer*

Let us visualise a given time-like curve in Minkowski 4-space. To guide our thoughts we think of it as the history of an arbitrary accelerated observer O on whose 'weltanschau' we are interested in. The given curve has parametric equations, say $\chi^\mu = \chi^\mu(\tau)$, relative to some set $\{\chi^\mu\}$ of space-time rectangular Cartesian coordinates. The parameter τ denotes O's proper time. We denote by u^μ and g^μ the 4-velocity and the 4-acceleration of O, respectively. Let us now introduce the Frenet-Serret tetrad $\{\gamma^\mu_{(\nu)}(\tau)\}$ on the curve and assume that O uses this tetrad as a local reference frame in space-time; i.e. we consider O as a Frenet-Serret observer. In effect, as was discussed in paper I, from the point of view of kinematics the Frenet-Serret tetrad is a particularly interesting object, for it affords the space-time representation of the comoving local frame pertaining to some arbitrarily moving (although not arbitrarily spinning) observer. Indeed, once the motion of O is given, that is, once the time-like world-line is given, the spin of the Frenet-Serret *triad* $\{\gamma^{\mu}_{(i)}\}$ is well defined along the curve via the Frenet-Serret formulas.

In order to understand the behaviour of the Frenet-Serret observer, it is useful to introduce an instantaneous comoving inertial observer. We denote by $O_0(\tau)$ that inertial observer who, at O's proper time τ, has the same space-time position and the same 4-velocity, relative to $\{\chi^{\mu}\}$, as O has; i.e. the history of $O_0(\tau)$ is precisely that straight world-line which is tangent to O at the event $\chi^{\mu}(\tau)$. This line is given by

$$\chi_0{}^{\mu}(\tau_0, \tau) = \chi^{\mu}(\tau) + (\tau_0 - \tau)u^{\mu}(\tau) \tag{2.1}$$

where τ_0 is O's proper time. The proper time τ of observer O appears in $\chi_0{}^{\mu}(\tau_0, \tau)$ as a parameter labelling, which one of the infinitely many comoving free observers we are considering, for, clearly, there is a different one for each value of τ. We see that, when $\tau_0 = \tau$, both observers are in instantaneous coincidence, have the same 4-velocity, and, moreover, for all values of τ_0, we have $g_0{}^{\mu} = (\partial^2\chi^{\mu}/\partial\tau_0{}^2) = 0$, as required.

Next we consider an orthonormal tetrad $\{\alpha^{\mu}_{(\nu)}\}$ attached to O_0's world-line. Without lost of generality, we assume that when $\tau_0 = \tau$ the inertial tetrad $\{\alpha^{\mu}_{(\nu)}\}$ and the Frenet-Serret tetrad $\{\gamma^{\mu}_{(\nu)}\}$ are instantaneously codirectional. Thus, these tetrads are the solutions of the following initial value problems:

$$(\partial/\partial\tau_0)\alpha^{\mu}_{(\nu)} = 0 \tag{2.2}$$

$$(d/d\tau)\gamma^{\mu}_{(\nu)} = C^{(\lambda)}_{(\nu)}(\tau)\gamma^{\mu}_{(\lambda)}(\tau) \tag{2.3}$$

$$\alpha^{\mu}_{(\nu)} = \gamma^{\mu}_{(\nu)}(\tau) \tag{2.4}$$

$$\gamma^{\mu}_{(0)}(\tau) = c^{-1}u^{\mu}(\tau) \tag{2.5}$$

The curvature matrix elements $C^{(\lambda)}_{(\nu)}$, namely, the Frenet-Serret formulas, are given explicitly in paper I; cf equation (3.1) of that paper. It must be borne in mind that both reference tetrads are in coincidence only at the event $\chi^{\mu}(\tau)$, for their laws of motion are quite different. For the sake of having a handy notation (and saving a lot of wording) we define

$$(\partial/\partial\tau_0)\gamma^{\mu}_{(\nu)}(\tau) = \lim_{\tau_0\to\tau}(\partial/\partial\tau_0)\alpha^{\mu}_{(\nu)} = 0 \tag{2.6}$$

while

$$\dot{\gamma}^{\mu}_{(\nu)}(\tau) = (d/d\tau)\gamma^{\mu}_{(\nu)}(\tau) \tag{2.7}$$

This notation automatically takes care of the different behaviour of both tetrads, and throws light on what case we are considering. With the above remark there is no danger of confusion and we shall write $\gamma^{\mu}_{(\nu)}(\tau)$, instead of $\alpha^{\mu}_{(\nu)}$, henceforth.

Some useful purely formal manipulations follow. As is well known, any tensor may be resolved into components along an orthonormal tetrad. Thus, for instance, given any vector field $V^\mu(\chi)$, we define

$$V^{(\nu)}(\tau_0, \tau) = \gamma_\mu^{(\nu)}(\tau) V^\mu [\chi_0(\tau_0, \tau)] \qquad (2.8)$$

along the world-line $O_0(\tau)$, and also

$$V^{(\nu)}(\tau) = \gamma_\mu^{(\nu)}(\tau) V^\mu [\chi(\tau)] \qquad (2.9)$$

along the world-line O. Of course, we want to interpret the inertial *triad* $\{\alpha_{(i)}^\mu\} = \{\gamma_{(i)}^\mu(\tau)\}$ as a frame supporting a system of rectangular Cartesian coordinates spanned over the physical 3-space of the inertial observer O_0. As a matter of fact, we want to consider the space components $V^{(i)}(\tau_0, \tau)$, $i = 1, 2, 3$, of equation (2.8), as the Cartesian components of a 3-vector $\mathbf{V}(\tau_0, \tau)$ belonging to Euclidean 3-space of observer $O_0(\tau)$, at time τ_0. In other words, we want to be able to write the usual linear combination

$$\mathbf{V}(\tau_0, \tau) = \boldsymbol{\Upsilon}_{(i)}(\tau) V^{(i)}(\tau_0, \tau) \qquad (2.10)$$

where $\{\boldsymbol{\Upsilon}_{(i)}(\tau)\}$ denotes a Cartesian basis spanned over $O_0(\tau)$'s space. Clearly, this basis remains fixed from the point of view of the comoving free observer; i.e. we require

$$(\partial/\partial\tau_0)\boldsymbol{\Upsilon}_{(i)}(\tau) = \mathbf{0} \qquad (2.11)$$

Furthermore, for the sake of handiness we shall assume it is a right-handed orthonormal basis. Thus, we introduce the usual 'dot' and 'cross' products for 3-vectors, such that

$$\boldsymbol{\Upsilon}_{(i)} \cdot \boldsymbol{\Upsilon}_{(j)} = \delta_{(i)(j)} \qquad (2.12)$$

and

$$\boldsymbol{\Upsilon}_{(i)} \times \boldsymbol{\Upsilon}_{(j)} = \epsilon_{(i)(j)(k)} \boldsymbol{\Upsilon}_{(k)} \qquad (2.13)$$

From equations (2.8) and (2.10), we observe that a *projection operator* can be introduced in the 'vierbein' formalism. This we do by defining the object

$$\boldsymbol{\Upsilon}_\mu(\tau) = \gamma_{(i)} \gamma_\mu^{(i)}(\tau) \qquad (2.14)$$

so, that equation (2.10) can be written as

$$\mathbf{V}(\tau_0, \tau) = \boldsymbol{\Upsilon}_\mu(\tau) V^\mu [\chi_0(\tau_0, \tau)] \qquad (2.15)$$

This operator is able to produce 3-vectors, belonging to the physical space of $O_0(\tau)$, at proper time τ_0, out of 4-vectors defined on the world-line of $O_0(\tau)$. It is a mixed object: it behaves as a 3-vector under space rotations of the Cartesian basis $\{\boldsymbol{\Upsilon}_{(i)}(\tau)\}$, and as a 4-vector under Lorentz transformations of the space-time system of coordinates $\{\chi^\mu\}$. The introduction of this projection operator turns out to be very helpful, for it brings the kinematics of the tetrad formalism to an intuitive compact form, allowing the introduction of the usual notation of ordinary vector analysis. For instance, if we adopt the representation $\gamma_{(1)}(\tau) = (1, 0, 0)$, $\gamma_{(2)}(\tau) = (0, 1, 0)$, $\gamma_{(3)}(\tau) = (0, 0, 1)$, for the Cartesian basis, then we have the simple representation $\boldsymbol{\Upsilon}_\mu(\tau) = (\gamma_\mu^{(1)}(\tau), \gamma_\mu^{(2)}(\tau), \gamma_\mu^{(3)}(\tau))$ for the projection operator. The algebra of $\boldsymbol{\Upsilon}_\mu$ is quite simple. Some interesting properties follow.

First, we observe that when projecting the 4-velocity of $O_0(\tau)$ itself, we get

$$\boldsymbol{\Upsilon}_\mu(\tau)u^\mu(\tau) = \mathbf{0} \tag{2.16}$$

so that, as a 4-vector, the projection operator is always orthogonal to the 4-velocity of the observer to whom it belongs. Next, let us consider the 4-tensor obtained by forming the 3-scalar $\boldsymbol{\Upsilon}_\mu \,.\boldsymbol{\Upsilon}_\nu$. One easily proves that

$$\boldsymbol{\Upsilon}_\mu \,.\boldsymbol{\Upsilon}_\nu = \gamma_\mu^{(0)}(\tau)\gamma_\nu^{(0)}(\tau) - \eta_{\mu\nu} = -\gamma_{\mu\nu}^{(0)}(\tau) \tag{2.17}$$

where $\gamma_{\mu\nu}^{(0)}(\tau)$ is the well-known projection tensor orthogonal to the 4-velocity $u^\mu(\tau) = c\gamma_{(0)}^\mu(\tau)$, and where $\eta_{\mu\nu}$ denotes the Minkowski metric. (We are using signature (−2), to be sure.) Thus, the projection operator corresponds to a factorisation (in the sense of the 3-scalar product) of the usual projection tensor. Another property we will need presently is the following 3-tensor (or dyadic) obtained from $\boldsymbol{\Upsilon}_\mu(\tau)$:

$$\boldsymbol{\Upsilon}_\mu(\tau)\boldsymbol{\Upsilon}^\mu(\tau) = -\delta_{(i)(j)}\boldsymbol{\Upsilon}_{(i)}(\tau)\boldsymbol{\Upsilon}_{(j)}(\tau) \tag{2.18}$$

From equations (2.6) and (2.11), we see that, as a 4-vector, the projection operator is parallel transported along the world-line $O_0(\tau)$, i.e. we have

$$(\partial/\partial\tau_0)\boldsymbol{\Upsilon}_\mu(\tau) = 0 \tag{2.19}$$

Finally, we observe that equation (2.14) can be inverted, to read

$$\boldsymbol{\Upsilon}_{(i)}(\tau) = \boldsymbol{\Upsilon}_\mu(\tau)\gamma_{(i)}^\mu(\tau) \tag{2.20}$$

This last equation amounts to adopt a more abstract, although equivalent, point of view: We consider the projection operator as a fundamental geometric object, and then the Cartesian basis spanning the physical space of the observer becomes just the projection of the triad on this space.

3. *Motion of the Frenet-Serret Observer Relative to the Comoving Inertial Frame*

As an instance of how the projection technique works, in this section we discuss the motion of the accelerated observer as seen by the instantaneous comoving free observer. Although the results are well known from elementary relativistic kinematics, we will briefly discuss the issue here for the sake of illustrating the use of the shorthand tool of the $\boldsymbol{\Upsilon}_\mu$ operator, while presenting some conclusions needed in the forthcoming section.

Let now τ' be any instant of O's proper time. We define, quite generally, the 4-vector of relative position in space-time:

$$\Delta\chi^\mu(\tau', \tau_0, \tau) = \chi^\mu(\tau') - \chi_0{}^\mu(\tau_0, \tau) \tag{3.1}$$

where τ_0 is any instant of proper time of O_0. In this equation we are considering two instants of O's proper time; namely, τ which is considered as fixed, and τ' which plays the role of the proper time variable. In particular, given τ and τ_0, we think over that value of τ' for which the relative position vector is orthogonal to the 4-velocity of $O_0(\tau)$; that is

$$u_\mu(\tau)\Delta\chi^\mu(\tau', \tau_0, \tau) = 0 \tag{3.2}$$

Here we have an equation for expressing this particular value for τ' as a function of τ_0 and τ; say $\tau' = \rho(\tau_0, \tau)$. After a little thought, we see that we must have $\lim_{\tau_0 \to \tau} \rho(\tau_0, \tau) = \tau$. So we define the *orthogonal position 4-vector* as

$$r^\mu(\tau_0, \tau) = \Delta\chi^\mu(\rho, \tau_0, \tau) \tag{3.3}$$

It is helpful to realise the meaning of these space-time position vectors $\Delta\chi^\mu$ and r^μ from the point of view of the usual parametric descriptions of time-like curves in special relativity. In effect, for given τ_0 and τ, the 4-vector $\Delta\chi^\mu$ represents the history of O relative to the inertial observer $O_0(\tau)$ as the proper time of O elapses; while, since

$$u_\mu(\tau)r^\mu(\tau_0, \tau) = 0 \tag{3.4}$$

for given τ, the 4-vector r^μ represents this same history as the proper time τ_0 of $O_0(\tau)$ elapses. Hence, both 4-vectors correspond to different parametrisations of the same curve relative to $O_0(\tau)$'s frame: $\Delta\chi^\mu$ gives the parametric equations in terms of proper time along the curve as parameter, while r^μ uses the coordinate time as parameter (for the proper time of an inertial observer is usually taken as his time coordinate).

We now project r^μ onto the physical space of O_0; i.e. we define

$$\mathbf{r}(\tau_0, \tau) = \boldsymbol{\Upsilon}_\mu(\tau)r^\mu(\tau_0, \tau) \tag{3.5}$$

This 3-vector represents the path of the accelerated observer in the 3-space of that free observer who, when $\tau_0 = \tau$, is comoving and coinciding with him. Again, in equation (3.5), τ_0 is the time variable, while τ figures as a fixed parameter. We can say that $\mathbf{r}$ gives the position O 'now', from the point of view of $O_0(\tau)$. As τ_0 elapses, O moves relative to $O_0(\tau)$ with a 3-velocity given by

$$\mathbf{v}(\tau_0, \tau) = \boldsymbol{\Upsilon}_\mu(\tau)(\partial/\partial\tau_0)r^\mu(\tau_0, \tau) \tag{3.6}$$

where equation (2.6) has been used. Therefore, from equations (2.1), (2.16), (3.1) and (3.3), one readily obtains

$$\mathbf{v}(\tau_0, \tau) = (\partial\rho/\partial\tau_0)\boldsymbol{\Upsilon}_\mu(\tau)u^\mu(\rho) \tag{3.7}$$

We observe, as is well known from relativistic kinematics, that the projection of the 4-velocity, i.e. $\boldsymbol{\Upsilon}_\mu(\tau)u^\mu(\rho)$, is not the 3-velocity The meaning of the relativistic factor $\partial\rho/\partial\tau_0$ is well known: it corresponds to the effect of *time dilation* between both observers. At any rate, it is easy to compute this factor in the present formalism. This we do, if only for the sake of showing the handiness of the projection approach. If we square equation (3.7), we get

$$\mathbf{v}^2 = (\partial\rho/\partial\tau_0)^2\{[\gamma_\mu^{(0)}(\tau)u^\mu(\rho)]^2 - c^2\} \tag{3.8}$$

On the other hand, equation (3.4) holds for all τ_0, and therefore, upon partial differentiation with respect to τ_0, it gives

$$u_\mu(\tau)\{(\partial\rho/\partial\tau_0)u^\mu(\rho) - u^\mu(\tau)\} = 0 \tag{3.9}$$

Thus, from equations (3.8) and (3.9), we obtain the well-known expression:

$$\partial\rho/\partial\tau_0 = [1 - c^{-2}\mathbf{v}^2(\tau_0, \tau)]^{1/2} \tag{3.10}$$

We next calculate, using the same method, the 3-acceleration of O relative to O_0, presenting the discussion in a very sketchy way. The physical 3-acceleration is, of course,

$$\begin{aligned}\mathbf{a}(\tau_0, \tau) &= (\partial/\partial\tau_0)\mathbf{v}(\tau_0, \tau)\\ &= \boldsymbol{\Upsilon}_\mu(\tau)(\partial/\partial\tau_0)^2 r^\mu(\tau_0, \tau)\end{aligned} \tag{3.11}$$

where

$$(\partial/\partial\tau_0)^2 r^\mu(\tau_0, \tau) = (\partial\rho/\partial\tau_0)^2 g^\mu(\rho) - c^{-2}(\partial\rho/\partial\tau_0)^{-1}\mathbf{v}\,.\,\mathbf{a}u^\mu(\rho) \tag{3.12}$$

Hence, if we introduce the projection of the 4-acceleration, say

$$\mathbf{g}(\tau_0, \tau) = \boldsymbol{\Upsilon}_\mu(\tau)g^\mu(\rho) \tag{3.13}$$

we get

$$\mathbf{a}(\tau_0, \tau) = (\partial\rho/\partial\tau_0)^2\mathbf{g} - \frac{1}{c^2}\left(\frac{\partial\rho}{\partial\tau_0}\right)^{-2}(\mathbf{v}\,.\,\mathbf{g})\mathbf{v} \tag{3.14}$$

and thus, after some manipulation, we obtain

$$\mathbf{g}(\tau_0, \tau) = \left(\frac{\partial\rho}{\partial\tau_0}\right)^{-4}\left[\mathbf{a} + \frac{1}{c^2}\mathbf{v}\times(\mathbf{v}\times\mathbf{a})\right] \tag{3.15}$$

This is the well-known formula for the space components of the 4-acceleration relative to an inertial frame. If we now define

$$\mathbf{g}(\tau) = \boldsymbol{\Upsilon}_\mu(\tau)g^\mu(\tau) \tag{3.16}$$

we see that

$$\mathbf{g}(\tau) = \lim_{\tau_0\to\tau}\mathbf{g}(\tau_0, \tau) = \lim_{\tau_0\to\tau}\mathbf{a}(\tau_0, \tau) \tag{3.17}$$

so that $\mathbf{g}(\tau)$ represents the physical 3-acceleration of O relative to $O_0(\tau)$ at the instant $\tau_0 = \tau$ when both observers are coinciding at relative rest.

Finally, let us study the motion of the Frenet-Serret triad $\{\gamma^{\mu}_{(i)}(\tau)\}$ from the point of view of $O_0(\tau)$. We define, for any value of τ_0,

$$\boldsymbol{\Upsilon}_{(i)}(\tau_0, \tau) = \boldsymbol{\Upsilon}_{\mu}(\tau)\gamma^{\mu}_{(i)}(\rho) \tag{3.18}$$

for $i = 1, 2, 3$. These 3-vectors represent the components of the Frenet-Serret triad in the physical space of $O_0(\tau)$, at time τ_0. We have, in general

$$\boldsymbol{\Upsilon}_{(i)} \cdot \boldsymbol{\Upsilon}_{(j)} = \delta_{(i)(j)} + \gamma^{(0)}_{\mu}(\tau)\gamma^{(0)}_{\nu}(\tau)\gamma^{\mu}_{(i)}(\rho)\gamma^{\nu}_{(j)}(\rho) \tag{3.19}$$

so that this moving triad does not look orthogonal to $O_0(\tau)$, unless when $\tau_0 = \tau$. We next define

$$\boldsymbol{\Upsilon}_{(i)}(\tau) = \lim_{\tau_0 \to \tau} \boldsymbol{\Upsilon}_{(i)}(\tau_0, \tau) = \boldsymbol{\Upsilon}_{\mu}(\tau)\gamma^{\mu}_{(i)}(\tau) \tag{3.20}$$

and

$$(\partial/\partial\tau_0)\boldsymbol{\Upsilon}_{(i)}(\tau) = \lim_{\tau_0 \to \tau} (\partial/\partial\tau_0)\boldsymbol{\Upsilon}_{(i)}(\tau_0, \tau) \tag{3.21}$$

We observe that the basis $\{\boldsymbol{\Upsilon}_{(i)}(\tau)\}$ corresponds precisely to the local frame used by O which, at time $\tau_0 = \tau$, is instantaneously codirectional with the frame used by O_0, although moving relative to it. Indeed, we have, according to equations (3.18) and (3.21),

$$(\partial/\partial\tau_0)\boldsymbol{\Upsilon}_{(i)}(\tau) = \lim_{\tau_0 \to \tau} (\partial\rho/\partial\tau_0)\gamma_{\mu}(\tau)\dot{\gamma}^{\mu}_{(i)}(\rho) \tag{3.22}$$

But, from the Frenet-Serret formulas, one readily obtains

$$\dot{\gamma}^{\mu}_{(i)} = \delta^{(1)}_{(i)} C_{(1)}\gamma^{\mu}_{(0)} + \epsilon_{(i)(j)(k)}\omega^{(k)}\gamma^{(j)\mu} \tag{3.23}$$

where $\epsilon_{(i)(j)(k)}$ is the three-dimensional permutation symbol, and where we have introduced, for later convenience, the quantities

$$\omega^{(1)} = -C_{(3)}, \qquad \omega^{(2)} = 0, \qquad \omega^{(3)} = -C_{(2)} \tag{3.24}$$

with $C_{(i)}$, $i = 1, 2, 3$, the three curvatures of the curve. Therefore, defining the rotation 3-vector

$$\boldsymbol{\omega}(\tau_0, \tau) = \boldsymbol{\Upsilon}_{(i)}(\tau)\omega^{(i)}(\rho) \tag{3.25}$$

and, in particular,

$$\boldsymbol{\omega}(\tau) = \lim_{\tau_0 \to \tau} \boldsymbol{\omega}(\tau_0, \tau) = \boldsymbol{\Upsilon}_{(i)}(\tau)\omega^{(i)}(\tau) \tag{3.26}$$

taking into account equations (2.16), (3.10) and (3.23), we obtain

$$(\partial/\partial\tau_0)\boldsymbol{\Upsilon}_{(i)}(\tau) = \boldsymbol{\omega}(\tau) \times \boldsymbol{\Upsilon}_{(i)}(\tau) \tag{3.27}$$

namely, the well-known result from elementary kinematics.

4. *Physical Identification of the Normals and Curvatures*

After these preliminaries, we now tackle the problem of the description of the normals and curvatures of the time-like curve O in terms of physical observables. From the Frenet-Serret formulas it is immediate that

$$\gamma^{\mu}_{(1)} = |g|^{-1} g^{\mu} \tag{4.1}$$

and therefore,

$$C_{(1)} = c^{-1}|g| \tag{4.2}$$

with $|g|$ denoting the Minkowskian norm of the 4-acceleration, i.e.

$$|g|^2 = -g_{\mu}g^{\mu} \tag{4.3}$$

Thus, we know $\gamma^{\mu}_{(1)}$ and $C_{(1)}$ in terms of g^{μ}. Furthermore, since

$$(d/d\tau)|g| = -\dot{g}_{\mu}\gamma^{\mu}_{(1)} \tag{4.4}$$

we obtain

$$\dot{g}^{\mu} = cC^{2}_{(1)}\gamma^{\mu}_{(0)} - \dot{g}_{\nu}\gamma^{\nu}_{(1)}\gamma^{\mu}_{(1)} + cC_{(1)}C_{(2)}\gamma^{\mu}_{(2)} \tag{4.5}$$

If we now introduce the projection tensor $\gamma^{\mu\nu}_{(1)}$, given by

$$\gamma^{\mu\nu}_{(1)} = \eta^{\mu\nu} + \gamma^{\mu}_{(1)}\gamma^{\nu}_{(1)} \tag{4.6}$$

we may define the (01)-projector as the contracted product of $\gamma^{\mu\nu}_{(0)}$ and $\gamma^{\mu\nu}_{(1)}$, namely

$$\gamma^{\mu\nu}_{(01)} = \gamma^{\mu\lambda}_{(0)}\gamma^{\nu}_{(1)\lambda} = \gamma^{\mu\nu}_{(0)} + \gamma^{\mu\nu}_{(1)} - \eta^{\mu\nu} \tag{4.7}$$

which is the projector onto the two-dimensional flat subspace orthogonal to $\gamma^{\mu}_{(0)}$ and $\gamma^{\mu}_{(1)}$, simultaneously. Hence, projecting $\dot{g}^{\mu}$ onto (01), only the $\gamma^{\mu}_{(2)}$ term survives:

$$\gamma^{\mu\nu}_{(01)}\dot{g}_{\nu} = cC_{(1)}C_{(2)}\gamma^{\mu}_{(2)} \tag{4.8}$$

and, hence

$$\gamma^{\mu\nu}_{(01)}\dot{g}_{\mu}\dot{g}_{\nu} = -c^2C^{2}_{(1)}C^{2}_{(2)} \tag{4.9}$$

Thus, having found $\gamma^{\mu}_{(1)}$ and $C_{(1)}$ in terms of g^{μ}, we can find $\gamma^{\mu}_{(2)}$ and $C_{(2)}$ in terms of g^{μ} and $\dot{g}^{\mu}$. We could continue the analysis along these lines; however, the results become rather cumbersome and are not very illuminating for the purposes of physical identification. It is here that the $\boldsymbol{\Upsilon}_{\mu}$ projection technique will give us some benefit.

Using the factorisation of the projector tensor $\gamma^{\mu\nu}_{(0)}$, we have

$$\gamma^{\mu\nu}_{(01)}\dot{g}_{\nu} = -\gamma^{\mu\lambda}_{(1)}\boldsymbol{\Upsilon}_{\lambda}.\boldsymbol{\Upsilon}_{\nu}\dot{g}^{\nu} \tag{4.10}$$

If we now define the limit

$$\dot{\mathbf{g}}_0(\tau) = (\partial/\partial\tau_0)\mathbf{g}(\tau) = \lim_{\tau_0\to\tau}(\partial/\partial\tau_0)\mathbf{g}(\tau_0,\tau) \tag{4.11}$$

we easily find

$$\dot{\mathbf{g}}_0(\tau) = \boldsymbol{\Upsilon}_\mu(\tau)\dot{g}^\mu(\tau) \tag{4.12}$$

Clearly, $\dot{\mathbf{g}}_0(\tau)$ represents the time rate of change of the physical acceleration of O, from the point of view of $O_0(\tau)$, at the very instant of coincidence $\tau_0 = \tau$. So equation (4.8) can be written as

$$-(\eta^{\mu\lambda} + \gamma^\mu_{(1)}\gamma^\lambda_{(1)})\,\boldsymbol{\Upsilon}_\lambda \cdot \dot{\mathbf{g}}_0 = cC_{(1)}C_{(2)}\gamma^\mu_{(2)} \tag{4.13}$$

and therefore, contracting this equation with $\gamma_{(2)\mu}$, one obtains

$$\boldsymbol{\Upsilon}_{(2)} \cdot \dot{\mathbf{g}}_0 = cC_{(1)}C_{(2)} \tag{4.14}$$

Furthermore, equations (4.13) and (4.14), we also get

$$\boldsymbol{\Upsilon}^\mu \cdot \dot{\mathbf{g}}_0 + \gamma^\mu_{(1)}\boldsymbol{\Upsilon}_{(1)} \cdot \dot{\mathbf{g}}_0 = -\gamma^\mu_{(2)}\boldsymbol{\Upsilon}_{(2)} \cdot \dot{\mathbf{g}}_0 \tag{4.15}$$

and hence, using the property stated in equation (2.18), the following equation results:

$$\dot{\mathbf{g}}_0 = g_0^{(1)}\boldsymbol{\Upsilon}_{(1)} + \dot{g}_0^{(2)}\boldsymbol{\Upsilon}_{(2)} \tag{4.16}$$

where, clearly, we have written

$$\dot{g}_0^{(i)} = \boldsymbol{\Upsilon}_{(i)} \cdot \dot{\mathbf{g}}_0 \tag{4.17}$$

Next, from equations (4.1) and (4.2), we easily obtain

$$\boldsymbol{\Upsilon}_\mu\gamma^\mu_{(1)} = |g|^{-1}\boldsymbol{\Upsilon}_\mu g^\mu = |g|^{-1}\mathbf{g} = \boldsymbol{\Upsilon}_{(1)} \tag{4.18}$$

i.e.

$$\mathbf{g} = |g|\boldsymbol{\Upsilon}_{(1)} \tag{4.19}$$

which gives the physical identification of $\boldsymbol{\Upsilon}_{(1)}$, namely, of the first normal, and

$$\mathbf{g} \cdot \mathbf{g} = |g|^2 = c^2C^2_{(1)} \tag{4.20}$$

which establishes the physical meaning of the first curvature $C_{(1)}$. Moreover, in equations (4.16) we have found

$$\dot{g}_0^{(3)} = \boldsymbol{\Upsilon}_{(3)} \cdot \mathbf{g} = 0 \tag{4.21}$$

and therefore, given the directions of $\mathbf{g}$ and $\dot{\mathbf{g}}_0$, the direction of $\boldsymbol{\Upsilon}_{(3)}$ is well determined (up to a sign). We take

$$\boldsymbol{\Upsilon}_{(3)} = |\mathbf{g} \times \dot{\mathbf{g}}_0|^{-1}\mathbf{g} \times \dot{\mathbf{g}}_0 \tag{4.22}$$

which states the identification of the third normal $\boldsymbol{\Upsilon}_{(3)}$ from the point of view of $O_0(\tau)$. But then the direction of $\boldsymbol{\Upsilon}_{(2)}$ has to be such as to have a right-handed Cartesian basis, i.e.

$$\boldsymbol{\Upsilon}_{(2)} = \boldsymbol{\Upsilon}_{(3)} \times \boldsymbol{\Upsilon}_{(1)} = (|\mathbf{g}||\mathbf{g} \times \dot{\mathbf{g}}_0|)^{-1}(\mathbf{g} \times \dot{\mathbf{g}}_0) \times \mathbf{g} \tag{4.23}$$

In this manner we have found the complete kinematical identification of the Frenet-Serret triad in the physical space of the instantaneous comoving inertial observer: equations (4.19), (4.22) and (4.23) are stating the physical meaning of the three space-time normals of a time-like world-line. Also the meaning of the first curvature is given in equation (4.20) as

$$C_{(1)} = c^{-1}|\mathbf{g}| \tag{4.24}$$

In order to physically interpret the two torsions (second and third curvatures) of the world-line, we now study the rotation of the Frenet-Serret triad $\{\gamma^{\mu}_{(i)}\}$ from the point of view of $O_0(\tau)$. This can be easily done by considering equation (3.27). We see the case $i = 1$ first. Using our result in equation (4.19), we have

$$(\partial/\partial\tau_0)\boldsymbol{\gamma}_{(1)} = \boldsymbol{\omega} \times \boldsymbol{\gamma}_{(1)} = (\partial/\partial\tau_0)|\mathbf{g}|^{-1} \qquad \mathbf{g} = \frac{\boldsymbol{\omega} \times \mathbf{g}}{|\mathbf{g}|} \tag{4.25}$$

which can be written as

$$\frac{|\mathbf{g} \times \dot{\mathbf{g}}_0|}{|\mathbf{g}|^2}\boldsymbol{\gamma}_{(2)} = \omega^{(3)}\boldsymbol{\gamma}_{(2)} - \omega^{(2)}\boldsymbol{\gamma}_{(3)} \tag{4.26}$$

Thus, we have

$$\omega^{(3)} = -C_{(2)} = |\mathbf{g}|^{-2}|\mathbf{g} \times \dot{\mathbf{g}}_0| \tag{4.27}$$

and also $\omega^{(2)} = 0$ (which we already knew). Next we consider the case $i = 3$. We have

$$(\partial/\partial\tau_0)\boldsymbol{\gamma}_{(3)} = \boldsymbol{\omega} \times \boldsymbol{\gamma}_{(3)} = (\partial/\partial\tau_0)\frac{\mathbf{g} \times \dot{\mathbf{g}}_0}{|\mathbf{g} \times \dot{\mathbf{g}}_0|} = \frac{\boldsymbol{\omega} \times (\mathbf{g} \times \dot{\mathbf{g}}_0)}{|\mathbf{g} \times \dot{\mathbf{g}}_0|} \tag{4.28}$$

After some manipulation this can be written as

$$\omega^{(1)} = -C_{(3)} = \frac{|\mathbf{g}|\mathbf{g} \cdot (\dot{\mathbf{g}}_0 \times \ddot{\mathbf{g}}_0)}{|\mathbf{g} \times \dot{\mathbf{g}}_0|^2} \tag{4.29}$$

Therefore, the rotation vector of the Frenet-Serret triad in the physical space of the comoving inertial observer is given by

$$\boldsymbol{\omega} = \frac{\mathbf{g} \times \dot{\mathbf{g}}_0}{|\mathbf{g}|^2} + \frac{[\mathbf{g} \cdot (\dot{\mathbf{g}}_0 \times \ddot{\mathbf{g}}_0)]\mathbf{g}}{|\mathbf{g} \times \dot{\mathbf{g}}_0|} \tag{4.30}$$

5. *Conclusion*

We emphasise the purely kinematic origin of the Frenet-Serret precession by noting that nothing has been said about the cause of the physical accelerations. We also remark on the fact that this effect takes place between two observers which are instantaneously at rest, one being accelerated relative to the other: the Cartesian frame of the accelerated observer precesses relative to the inertial comoving frame because, according to equations (4.19) and (4.22), the Frenet-Serret observer permanently orients his basis in the direction of the accelerations he feels. Thus, for instance, the moon evolves under Frenet-Serret transport in its motion around the earth.

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